Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 1, pp. 9–17, 2009

NONSTATIONARY AXISYMMETRIC PROBLEM OF ELECTROELASTICITY FOR A PIEZOCERAMIC CYLINDER WITH CIRCULAR POLARIZATION

D. A. Shlyakhin

UDC 539.3

This paper considers the problem of the propagation of forced electroelastic axisymmetric torsional waves in a hollow piezoceramic cylinder of finite dimensions, whose curvilinear electroded surfaces are subjected to shear stresses or an electrical potential. A new closed solution is constructed by expansion in vector eigenfunctions using the structural algorithm of finite integral transforms, which makes it possible to determine vibration eigenfrequencies, the stress–strain state of the element, and the potential and intensity of the induced electrical field.

Key words: *coupled problem, direct and inverse piezoeffects, cylinder of finite dimensions, axisymmetric dynamic loading.*

Introduction. The main element of a wide class of pulsed converters of energy is a piezoceramic cylinder of finite dimensions, whose operation is based on the coupling of mechanical and electrical stress fields. In the case of circular polarization of the piezomaterial, this effect is observed only during propagation of nonstationary axisymmetric torsional waves. Because of the complexity of this problem, most studies have been focused on electroelasticity problems for an infinite cylinder under harmonic action [1, 2] and free vibrations of a thick plate under various boundary-value conditions on electric fields [1]. Mention should also be made of the solution which is valid for heterogeneous crystals of the tetragonal symmetry class 422 with the curvilinear surfaces subjected to a dynamic load element in the form of an electrical potential or shear stresses [3].

The present paper considers the propagation of forced electroelastic axisymmetric torsional waves in a hollow piezoceramic cylinder of finite dimensions.

1. Formulation of the Problem. The hollow cylinder occupying a region $\{\Omega\colon\ a \leqslant r_* \leqslant b,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant z_* \leqslant h\}$ in the cylindrical coordinate system (r_*, θ, z_*) is a linearly elastic anisotropic body and is made of a piezoceramic material with induced circular polarization. The case is considered where the end planes of the element are not fixed and are free of electrical charges, and the inner and outer curvilinear electroded surfaces are subjected to shear stresses $\sigma_1^*(z_*, t_*)$ and $\sigma_2^*(z_*, t_*)$ and a potential $V^*(z_*, t_*)$.

In this formulation, the boundary-value problem models the work of piezoelements in devices based on the direct and inverse piezoeffects; in the first case, the mechanical action is transformed to the corresponding electrical signal [see the boundary conditions of type (a) and (b) on the curvilinear planes given below], and in the second case, electrical load is transformed to deformation [see type (c)]. In addition, different methods of measuring the induced electrical signal [4] are considered: in type (a), the radial surfaces are completely or partly electroded and are connected to a measuring device with high input impedance, which corresponds to no-load conditions (the absence of free electrical charges); in type (b), the completely electroded equipotential planes are coupled to a measuring device with low input impedance.

Samara State University of Architecture and Civil Engineering, Samara 443001; sgasu@sgasu.smr.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 1, pp. 12–21, January–February, 2009. Original article submitted December 5, 2007.

Generally, the differential equations of motion and electrostatics for a homogeneous elastic anisotropic medium in cylindrical coordinates are written as [1]

$$\frac{\partial \sigma_{r\theta}}{\partial r_*} + \frac{\partial \sigma_{z\theta}}{\partial z_*} + \frac{2}{r_*}\,\sigma_{r\theta} - \rho\,\frac{\partial^2 v^*}{\partial t_*^2} = 0,$$
$$\frac{\partial D_r}{\partial r_*} + \frac{D_r}{r_*} + \frac{\partial D_z}{\partial z_*} = 0. \tag{1.1}$$

In the case of circular polarization, the equations of state for a piezoceramic solid are given by the following equalities [1]:

$$\sigma_{r\theta} = C_{55}\Big(\frac{\partial v^*}{\partial r_*} - \frac{v^*}{r_*}\Big) - e_{15}E_r, \qquad \sigma_{z\theta} = C_{55}\,\frac{\partial v^*}{\partial z_*} - e_{15}E_z,$$
$$D_r = \varepsilon_{11}E_r + e_{15}\Big(\frac{\partial v^*}{\partial r_*} - \frac{v^*}{r_*}\Big), \qquad D_z = \varepsilon_{11}E_z + e_{15}\,\frac{\partial v^*}{\partial z_*}, \tag{1.2}$$
$$E_z = -\frac{\partial \varphi^*}{\partial z_*}, \qquad E_r = -\frac{\partial \varphi^*}{\partial r_*}.$$

In relations (1.1) and (1.2), t_* is time; $\sigma_{r\theta}(r_*, z_*, t_*)$ and $\sigma_{z\theta}(r_*, z_*, t_*)$ are the mechanical stress tensor components; $v^*(r_*, z_*, t_*)$ is the tangential component of the displacement vector; $D_r(r_*, z_*, t_*)$ and $D_z(r_*, z_*, t_*)$ and $E_r(r_*, z_*, t_*)$ and $E_z(r_*, z_*, t_*)$ are the components of the induction and intensity vectors, respectively; $\varphi^*(r_*, z_*, t_*)$ is the electrical-field potential; ρ, C_{55}, and e_{15} are the volumetric density, elastic modulus, and piezomodulus of the anisotropic electroelastic material, respectively, and ε_{11} is the dielectric permeability.

Substituting (1.2) into (1.1), for the dynamic problem of electroelasticity considered, we obtain the differential equations in dimensionless form

$$\nabla_1 v + \frac{\partial^2 v}{\partial z^2} + e_{15}\Big(\nabla_2 + \frac{\partial^2}{\partial z^2}\Big)\varphi - \frac{\partial^2 v}{\partial t^2} = 0,$$
$$e_{15}\Big(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\Big)v - C_{55}\varepsilon_{11}\Big(\nabla_3 + \frac{\partial^2}{\partial z^2}\Big)\varphi = 0, \tag{1.3}$$

the boundary conditions

$$z = 0, L: \qquad \sigma_{z\theta} = C_{55}\,\frac{\partial v}{\partial z} + e_{15}\,\frac{\partial \varphi}{\partial z} = 0, \qquad D_z = -C_{55}\varepsilon_{11}\,\frac{\partial \varphi}{\partial z} + e_{15}\,\frac{\partial v}{\partial z} = 0; \tag{1.4}$$

$r = 1, k$:

(a) $$\sigma_{r\theta}\Big|_{r=1} = \frac{\partial v}{\partial r} - \frac{v}{r} + e_{15}\,\frac{\partial \varphi}{\partial r} = \sigma_1(z,t), \qquad \sigma_{r\theta}\Big|_{r=k} = \frac{\partial v}{\partial r} - \frac{v}{r} + e_{15}\,\frac{\partial \varphi}{\partial r} = \sigma_2(z,t),$$
$$D_r = -C_{55}\varepsilon_{11}\,\frac{\partial \varphi}{\partial r} + e_{15}\Big(\frac{\partial v}{\partial r} - \frac{v}{r}\Big) = 0; \tag{1.5a}$$

(b) $$\sigma_{r\theta}\Big|_{r=1} = \frac{\partial v}{\partial r} - \frac{v}{r} + e_{15}\,\frac{\partial \varphi}{\partial r} = \sigma_1(z,t), \qquad \sigma_{r\theta}\Big|_{r=k} = \frac{\partial v}{\partial r} - \frac{v}{r} + e_{15}\,\frac{\partial \varphi}{\partial r} = \sigma_2(z,t),$$
$$E_z = -\frac{\partial \varphi}{\partial z} = 0; \tag{1.5b}$$

(c) $$\sigma_{r\theta}\Big|_{r=1} = \frac{\partial v}{\partial r} - \frac{v}{r} + e_{15}\,\frac{\partial \varphi}{\partial r} = 0, \qquad v(k,z,t) = 0,$$
$$\varphi(1,z,t) = V(z,t), \qquad \varphi(k,z,t) = -V(z,t), \tag{1.5c}$$

and the initial conditions

$$t = 0: \qquad v(r,z,0) = v_0(r,z), \qquad \dot v(r,z,0) = \dot v_0(r,z). \tag{1.6}$$

In (1.3)–(1.6), $\{v, r, z, L, k\} = \{v^*, r_*, z_*, h, a\}/b$, $\{\varphi, V, \sigma_1, \sigma_2\} = \{\varphi^*, V^*, \sigma_1^*, \sigma_2^*\}/(bC_{55})$, $t = t_* b^{-1}\sqrt{C_{55}/\rho}$, v_0 and $\dot{v}_0$ are the tangential displacements and their velocities known at the initial time; the point denotes differentiation with respect to time;

$$\nabla_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}, \qquad \nabla_2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}, \qquad \nabla_3 = \nabla_1 + \frac{1}{r^2}.$$

In the investigation of the direct piezoeffect, the axisymmetric load is self-balanced if the following condition is satisfied:

$$\int_k^1 \sigma_1(z,t)\,dz + k\int_k^1 \sigma_2(z,t)\,dz = 0, \tag{1.7}$$

and in the investigation of the inverse piezoeffect, the inner radial surfaces are assumed, for definiteness, to be fixed.

Relations (1.3)–(1.7) represent the mathematical formulation of the examined initial-boundary-value problem of electroelasticity.

2. Construction of the General Solution. Applying the Fourier cosine transform with finite limits along the variable z to the initial-boundary-value problem (1.3)–(1.6), in the space of images we obtain the system of differential equations

$$\begin{gathered}\nabla_1 v_c - j_n^2 v_c + e_{15}(\nabla_2 - j_n^2)\varphi_c - \frac{\partial^2 v_c}{\partial t^2} = 0,\\ e_{15}\Big(\frac{\partial^2}{\partial r^2} - j_n^2\Big)v_c - C_{55}\varepsilon_{11}(\nabla_3 - j_n^2)\varphi_c = 0,\end{gathered} \tag{2.1}$$

and boundary conditions for $r = 1, k$:

(a) $$\begin{gathered}\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)v_c + e_{15}\frac{\partial\varphi_c}{\partial r} = N_{1c}\Big|_{r=1}, \qquad \Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)v_c + e_{15}\frac{\partial\varphi_c}{\partial r} = N_{2c}\Big|_{r=k},\\ -C_{55}\varepsilon_{11}\frac{\partial\varphi_c}{\partial r} + e_{15}\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)v_c = 0;\end{gathered} \tag{2.2a}$$

(b) $$\begin{gathered}\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)v_c + e_{15}\frac{\partial\varphi_c}{\partial r} = N_{1c}\Big|_{r=1}, \qquad \Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)v_c + e_{15}\frac{\partial\varphi_c}{\partial r} = N_{2c}\Big|_{r=k},\\ \varphi_c = 0;\end{gathered} \tag{2.2b}$$

(c) $$\begin{gathered}\Big(\frac{\partial v_c}{\partial r} - \frac{v_c}{r} + e_{15}\frac{\partial\varphi_c}{\partial r}\Big)\Big|_{r=1} = 0, \qquad v_c(k,n,t) = 0,\\ \varphi_c(1,n,t) = V_c(n,t), \qquad \varphi_c(k,n,t) = -V_c(n,t),\end{gathered} \tag{2.2c}$$

and the initial conditions

$$t = 0: \qquad v_c(r,n,0) = v_{0c}(r,n), \qquad \dot{v}_c(r,n,0) = \dot{v}_{0c}(r,n), \tag{2.3}$$

where

$$\begin{gathered}\{v_c(r,n,t), \varphi_c(r,n,t), v_{0c}(r,n), \dot{v}_{0c}(r,n), N_{1c}(1,n,t), N_{2c}(k,n,t), V_c(n,t)\}\\ = \int_0^L \{v(r,z,t), \varphi(r,z,t), v_0(r,z), \dot{v}_0(r,z), \sigma_1(1,z,t), \sigma_2(k,z,t), V(z,t)\}\cos(j_n z)\,dz,\\ j_n = n\pi/L \qquad (n = \overline{0,\infty}).\end{gathered}$$

Reducing the boundary-value conditions to homogeneous conditions in the initial-boundary-value problem (2.1)–(2.3), we represent the Fourier transforms v_c and φ_c in the form

$$v_c(r,n,t) = H_1(r,n,t) + p_c(r,n,t), \qquad \varphi_c(r,n,t) = H_2(r,n,t) + \chi_c(r,n,t), \tag{2.4}$$

where, for the boundary conditions of type (a), we have

$$H_1(r,n,t) = \frac{C_{55}\varepsilon_{11}e_{15}}{C_{55}\varepsilon_{11}+e_{15}^2}\,[f_1(r)N_{1c}+f_2(r)N_{2c}], \qquad H_2(r,n,t) = f_3(r)N_{1c}+f_4(r)N_{2c},$$

for the boundary conditions of type (b),

$$H_1(r,n,t) = f_1(r)N_{1c}+f_2(r)N_{2c}, \qquad H_2(r,n,t) = 0,$$

and for the boundary conditions of type (c),

$$H_1(r,n,t) = 0, \qquad H_2(r,n,t) = f_3(r)V_c.$$

Substituting (2.4) into (2.1)–(2.3) and taking into account that

$$\begin{gathered} f_1(1) = f_1(k) = f_2(1) = f_2(k) = f_1'(k) = f_2'(1) = 0, \\ f_1'(1) = f_2'(k) = e_{15}^{-1}, \qquad f_3'(1) = f_4'(k) = e_{15}/(C_{55}\varepsilon_{11}+e_{15}^2), \\ f_3'(k) = f_4'(1) = 0 \end{gathered} \tag{2.5a}$$

for the boundary conditions of type (a),

$$\begin{gathered} f_1(1) = f_1(k) = f_2(1) = f_2(k) = f_1'(k) = f_2'(1) = 0, \\ f_1'(1) = f_2'(k) = 1 \end{gathered} \tag{2.5b}$$

for the boundary conditions of type (b), and

$$f_3(1) = -f_3(k) = 1, \qquad f_3'(1) = 0 \tag{2.5c}$$

for the boundary conditions of type (c), we obtain the initial-boundary-value problem for the functions p_c and χ_c with homogeneous boundary conditions on the coordinate r (prime denotes differentiation with respect to the variable r). This problem includes the differential equations

$$\begin{gathered} \nabla_1 p_c - j_n^2 p_c + e_{15}(\nabla_2 - j_n^2)\chi_c - \frac{\partial^2 p_c}{\partial t^2} = B_{1c}, \\ e_{15}\Big(\frac{\partial^2}{\partial r^2} - j_n^2\Big)p_c - C_{55}\varepsilon_{11}(\nabla_3 - j_n^2)\chi_c = B_{2c}, \end{gathered} \tag{2.6}$$

the boundary conditions for $r = 1, k$

(a) $$\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)p_c = 0, \qquad \frac{\partial \chi_c}{\partial r} = 0; \tag{2.7a}$$

(b) $$\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)p_c + e_{15}\,\frac{\partial \chi_c}{\partial r} = 0, \qquad \chi_c = 0; \tag{2.7b}$$

(c) $$\Big(\frac{\partial p_c}{\partial r} - \frac{p_c}{r} + e_{15}\,\frac{\partial \chi_c}{\partial r}\Big)\Big|_{r=1} = 0, \qquad p_c(k,n,t) = 0, \qquad \chi_c = 0, \tag{2.7c}$$

and the initial conditions

$$t = 0\colon \qquad p_c(r,n,0) = p_{0c}(r,n), \qquad \dot p_c(r,n,0) = \dot p_{0c}(r,n), \tag{2.8}$$

where

$$B_{1c} = -\Big(\nabla_1 - j_n^2 - \frac{\partial^2}{\partial t^2}\Big)H_1 - e_{15}(\nabla_2 - j_n^2)H_2,$$

$$B_{2c} = -e_{15}\Big(\frac{\partial^2}{\partial r^2} - j_n^2\Big)H_1 + C_{55}\varepsilon_{11}(\nabla_3 - j_n^2)H_2,$$

$$p_{0c}(r,n) = v_{0c}(r,n) - H_1\Big|_{t=0}, \qquad \dot p_{0c}(r,n) = \dot v_{0c}(r,n) - \dot H_1\Big|_{t=0}.$$

For the boundary conditions of type (a) and (b), the functions $f_1(r), \ldots, f_4(r)$ are determined from the differential equations

$$f_1^{(\mathrm{IV})}(r) = f_2^{(\mathrm{IV})}(r) = 0, \qquad f_3''(r) = f_4''(r) = \mathrm{const}, \tag{2.9a}$$

and for the boundary conditions of type (c), they are obtained from the differential equation

$$f_3^{(\mathrm{IV})}(r) = 0. \tag{2.9b}$$

We shall solve the initial-boundary-value problem (2.6)–(2.8) for $p_c(r,n,t)$ and, $\chi_c(r,n,t)$ using the structural algorithm of the method of finite integral transforms [5]. In the interval $[k,1]$, we introduce a degenerate finite integral transform With the unknown components $K_1(\lambda_{in},r)$ and $K_2(\lambda_{in},r)$ of the vector function of the transform kernel:

$$G(\lambda_{in},n,t) = \int_k^1 p_c(r,n,t)K_1(\lambda_{in},r)r\,dr,$$

$$p_c(r,n,t) = \sum_{i=1}^{\infty} G(\lambda_{in},n,t)K_1(\lambda_{in},r)\|K_{in}\|^{-2}, \tag{2.10}$$

$$\chi_c(r,n,t) = \sum_{i=1}^{\infty} G(\lambda_{in},n,t)K_2(\lambda_{in},r)\|K_{in}\|^{-2}, \qquad \|K_{in}\|^2 = \int_k^1 K_1^2(\lambda_{in},r)r\,dr.$$

Here λ_{in} $(i=\overline{1,\infty})$ are positive parameters which form a countable set; $\|K_{in}\|$ is the norm of the vector function of the degenerate transform]. In this case, the circular frequencies of the axisymmetric vibrations of the cylinder ω_{in} are linked to λ_{in} by the relation

$$\omega_{in} = (\lambda_{in}/b)\sqrt{C_{55}/\rho}\,. \tag{2.11}$$

Transforming system (2.6) in accordance with the structural algorithm [5], we obtain a countable set of Cauchy problems

$$\ddot{G}(\lambda_{in},n,t) + \lambda_{in}^2 G(\lambda_{in},n,t) = -F(\lambda_{in},n,t) \qquad (n=\overline{0,\infty}, \quad i=\overline{1,\infty}\,); \tag{2.12}$$

$$G(\lambda_{in},n,0) = G_0(\lambda_{in},n), \qquad \dot{G}(\lambda_{in},n,t)\Big|_{t=0} = \dot{G}_0(\lambda_{in},n), \quad t=0 \tag{2.13}$$

for the transform $G(\lambda_{in},n,t)$ and a homogeneous boundary-value problem for the components K_1 and K_2 of the finite integral transform kernel, which includes the differential equations

$$\begin{aligned} &\nabla_1 K_1 + (\lambda_{in}^2 - j_n^2)K_1 + e_{15}(\nabla_2 - j_n^2)K_2 = 0, \\ &e_{15}\Big(\frac{d^2}{dr^2} - j_n^2\Big)K_1 - C_{55}\varepsilon_{11}(\nabla_3 - j_n^2)K_2 = 0 \end{aligned} \tag{2.14}$$

and the boundary conditions for $r = 1, k$:

(a) $$\Big(\frac{d}{dr} - \frac{1}{r}\Big)K_1 = 0, \qquad \frac{dK_2}{dr} = 0; \tag{2.15a}$$

(b) $$\Big(\frac{d}{dr} - \frac{1}{r}\Big)K_1 + e_{15}\frac{dK_2}{dr} = 0, \qquad K_2 = 0; \tag{2.15b}$$

(c) $$\Big(\frac{dK_1}{dr} - \frac{K_1}{r} + e_{15}\frac{dK_2}{dr}\Big)\Big|_{r=1} = 0, \qquad K_1(\lambda_{in},k) = 0, \qquad K_2 = 0. \tag{2.15c}$$

In Eqs. (2.12) and (2.13), we obtain

$$F(\lambda_{in},n,t) = \int_k^1 (B_{1c}K_1 + B_{2c}K_2)r\,dr,$$

$$G_0(\lambda_{in},n) = \int_k^1 p_{0c}K_1 r\,dr, \qquad \dot{G}_0(\lambda_{in},n) = \int_k^1 \dot{p}_{0c}K_1 r\,dr.$$

In view of (2.13), the solution of Eq. (2.12) is written as

$$G(\lambda_{in}, n, t) = G_0 \cos \lambda_{in} t + \dot{G}_0 \lambda_{in}^{-1} \sin \lambda_{in} t + \lambda_{in}^{-1} \int\limits_0^t F(\lambda_{in}, n, t) \sin \lambda_{in}(t-\tau)\, d\tau. \tag{2.16}$$

By using the new functions $R_1(\lambda_{in}, r)$ and $R_2(\lambda_{in}, r)$, which are linked to $K_1(\lambda_{in}, r)$ and $K_2(\lambda_{in}, r)$ by the relations

$$\begin{aligned} K_1(\lambda_{in}, r) &= -\nabla_4 R_1(\lambda_{in}, r) + \nabla_5 R_2(\lambda_{in}, r), \\ K_2(\lambda_{in}, r) &= -\frac{e_{15}}{C_{55}\varepsilon_{11}} \left[\nabla_4 R_1(\lambda_{in}, r) + \nabla_6 R_2(\lambda_{in}, r)\right], \end{aligned} \tag{2.17}$$

system (2.14) can be reduced to the following resolving differential equation for R_1 and R_2:

$$\nabla_7 R_1 + \nabla_8 R_2 = 0. \tag{2.18}$$

Here

$$\nabla_4 = r^2 \frac{d^3}{dr^3} - 3r \frac{d^2}{dr^2} + 3 \frac{d}{dr},$$

$$\nabla_5 = r^2 \frac{d^2}{dr^2} + 3r \frac{d}{dr} - j_n^2 r^2, \qquad \nabla_6 = -r^2 \frac{d^2}{dr^2} - 2r \frac{d}{dr} + 2 + j_n^2 r^2,$$

$$\nabla_7 = b_1 r^2 \frac{d^5}{dr^5} + (3b_1 + 1) r \frac{d^4}{dr^4} + (b_2 r^2 - 3b_1) \frac{d^3}{dr^3} + \left(3b_2 r - \frac{3}{r}\right) \frac{d^2}{dr^2} + \left(\frac{3}{r^2} - 3b_2\right) \frac{d}{dr},$$

$$\nabla_8 = -b_1 r^2 \frac{d^4}{dr^4} - 8b_1 r \frac{d^3}{dr^3} + [(b_2 + j_n^2 b_1) r^2 - 12 b_1] \frac{d^2}{dr^2} + (3b_2 + 5 j_n^2 b_1) r \frac{d}{dr} + j_n^2 (4b_1 + 1 - b_2 r^2),$$

$$b_1 = 1 + e_{15}^2/(C_{55}\varepsilon_{11}), \qquad b_2 = \lambda_{in}^2 + j_n^2 b_1.$$

The partial solution of the differential equation (2.18) is found by using the following representations of the functions $R_1(\lambda_{in}, r)$ and $R_2(\lambda_{in}, r)$:

$$\begin{aligned} R_1(\lambda_{in}, r) &= (P_{1in} r^3 + P_{2in} r^5 + P_{3in} r^7) J_0(\sigma_{in} r) + (P_{4in} r^4 + P_{5in} r^6) J_1(\sigma_{in} r), \\ R_2(\lambda_{in}, r) &= (P_{6in} r^2 + P_{7in} r^4 + P_{8in} r^6) J_0(\sigma_{in} r) + (r + P_{9in} r^3 + P_{10in} r^5 + P_{11in} r^7) J_1(\sigma_{in} r). \end{aligned} \tag{2.19}$$

Here $J_0(\,\cdot\,)$ and $J_1(\,\cdot\,)$ are Bessel function of the first kind of the zero and first order and σ_{in} and $P_{1in}, \ldots, P_{11in}$ are currently unknown coefficients.

Transforming equality (2.18) in view of (2.19) and setting all multipliers of the result equal to zero, we obtain the following system of inhomogeneous algebraic equations for the unknown coefficients $P_{1in}, \ldots, P_{11in}$ and the characteristic equation for σ_{in}:

$$\sigma_{in}^2 - 4\lambda_{in}^2 \sigma_{in} + 2\lambda_{in}^4 = 0. \tag{2.20}$$

Taking into account relations (2.17) and (2.19) and the fact that the roots of the characteristic equation (2.20) are positive, we obtain a partial solution of system (2.14):

$$\begin{aligned} K_1(\lambda_{in}, r) &= \eta_{in}(r) J_0(\sigma_{in} r) + \gamma_{in}(r) J_1(\sigma_{in} r), \\ K_2(\lambda_{in}, r) &= \mu_{in}(r) J_0(\sigma_{in} r) + \psi_{in}(r) J_1(\sigma_{in} r). \end{aligned} \tag{2.21}$$

The system of equations whose solution yields the coefficients $P_{1in}, \ldots, P_{11in}$ and the expressions for $\eta_{in}(r)$, $\gamma_{in}(r)$, $\mu_{in}(r)$, and $\psi_{in}(r)$ are not given in the present paper.

In view of (2.21) and the recursive relations for the Bessel functions of the first and second kind of [6], the fundamental solution (2.14) is written as

$$K_1(\lambda_{in}, r) = \sum_{j=1}^{2} D_{jin}[\eta_{jin}(r)J_0(\sigma_{jin}r) + \gamma_{jin}(r)J_1(\sigma_{jin}r)]$$

$$+ \sum_{j=1}^{2} D_{(2+j)in}[\eta_{(2+j)in}(r)Y_0(\sigma_{jin}r) + \gamma_{(2+j)in}(r)Y_1(\sigma_{jin}r)],$$

$$K_2(\lambda_{in}, r) = \sum_{j=1}^{2} D_{jin}[\mu_{jin}(r)J_0(\sigma_{jin}r) + \psi_{jin}(r)J_1(\sigma_{jin}r)] \tag{2.22}$$

$$+ \sum_{j=1}^{2} D_{(2+j)in}[\mu_{(2+j)in}(r)Y_0(\sigma_{jin}r) + \psi_{(2+j)in}(r)Y_1(\sigma_{jin}r)].$$

Here $Y_\mu(\,\cdot\,)$ are Bessel functions of the second kind of order μ ($\mu = 0.1$).

Substitution of (2.22) into boundary conditions (2.15) yields a homogeneous system of equations for the constants $D_{1in}, \ldots, D_{4in}$. Having found its nontrivial solutions, we obtain a transcendental equation for calculating the eigenvalues λ_{in}

$$\det[B_{mp}] = 0 \qquad (m = \overline{1,4}, \quad p = \overline{1,4}),$$

and expressions for the constants $D_{1in}, \ldots, D_{4in}$.

Without loss of generality, we set $D_{4in} = 1$. The remaining integration constants can be determined by solving the following system of inhomogeneous equations:

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} D_{1in} \\ D_{2in} \\ D_{3in} \end{bmatrix} = \begin{bmatrix} B_{14} \\ B_{24} \\ B_{34} \end{bmatrix}.$$

3. Computation Formulas. The final stage of the study is the determination of the functions $f_1(r), \ldots, f_4(r)$ in representations (2.4). Using the differential equations (2.9) and the corresponding boundary conditions (2.5), we have:

— for the boundary conditions of type (a),

$$f_1(r) = [e_{15}(k-1)^2(2k+1)]^{-1}[-r^3 + (2k+1)r^2 - k(k+2)r + k^2],$$

$$f_2(r) = [e_{15}(k-1)^2]^{-1}[r^3 - (k+2)r^2 + (2k+1)r - k],$$

$$f_3(r) = \frac{e_{15}}{(C_{55}\varepsilon_{11} + e_{15}^2)(1-k)}\,(0.5r^2 - kr), \qquad f_4(r) = \frac{e_{15}}{(C_{55}\varepsilon_{11} + e_{15}^2)(k-1)}\,(0.5r^2 - r);$$

— for the boundary conditions of type (b),

$$f_1(r) = -[(k-1)^2(2k+1)]^{-1}[-r^3 + (2k+1)r^2 - k(k+2)r + k^2],$$

$$f_2(r) = (k-1)^{-2}[r^3 - (k+2)r^2 + (2k+1)r - k];$$

— and for the boundary conditions of type (c),

$$f_3(r) = (k-1)^{-2}(-2r^2 + 4r + k^2 - 2k - 1).$$

Applying successively the inversion formulas (2.10) and the finite Fourier cosine transform formulas to transform (2.16) and taking into account (2.4), we obtain the following expansions for $v(r,z,t)$ and $\varphi(r,z,t)$:

$$v(r,z,t) = \sum_{n=0}^{\infty} \Omega^{-1}\Big[H_1(r,n,t) + \sum_{i=1}^{\infty} G(\lambda_{in},n,t)K_1(\lambda_{in}r)\|K_{in}\|^{-2}\Big]\cos(j_n z),$$

$$\varphi(r,z,t) = \sum_{n=0}^{\infty} \Omega^{-1}\Big[H_2(r,n,t) + \sum_{i=1}^{\infty} G(\lambda_{in},n,t)K_2(\lambda_{in}r)\|K_{in}\|^{-2}\Big]\cos(j_n z).$$

TABLE 1

Eigenvalues λ_{in} for the First Three Modes of Axisymmetric Torsional Waves of the Cylinder

i	λ_{in}							
	Boundary conditions of type (a)				Boundary conditions of type (b)			
	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 0$	$n = 1$	$n = 2$	$n = 3$
1	0.181	0.489	0.358	0.435	0.148	0.307	0.473	0.585
2	0.922	0.685	0.386	0.473	0.224	0.687	0.781	0.844
3	1.484	1.251	0.515	0.665	0.642	1.261	1.861	1.975

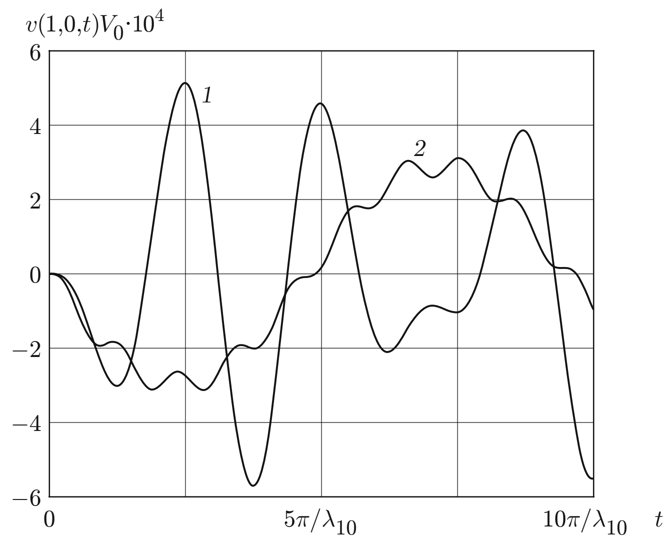

Fig. 1

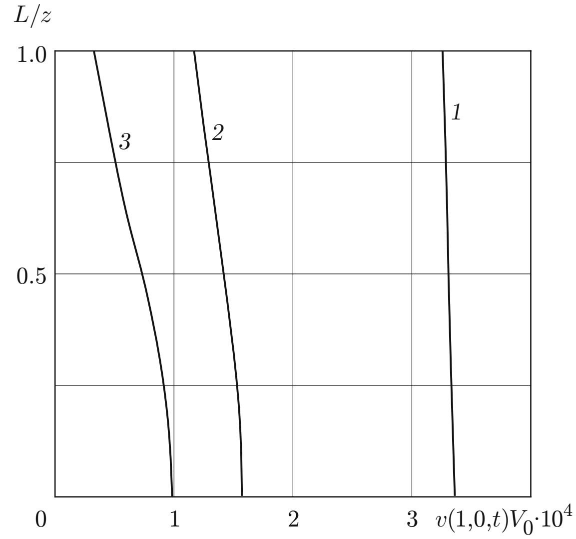

Fig. 2

Fig. 1. Tangential displacements of the outer curvilinear surface of the cylinder versus time for $\theta = 0.6\lambda_{10}$ (1) and $0.2\lambda_{10}$ (2).

Fig. 2. Variation in the relative displacements along the height of the cylinder for $a = 0.75b$ (1), $0.5b$ (2), and $0.25b$ (3).

Here

$$\Omega_n = \begin{cases} L, & n = 0, \\ L/2, & n \neq 0. \end{cases}$$

For the direct piezoeffect, the potential difference $Q(t_*)$ between the electroded radial planes of the piezoceramic cylinder is defined as follows [7]:

— for the boundary conditions of type (a),

$$Q(t_*) = Q(b, t_*) - Q(a, t_*);$$

— for the boundary conditions of type (b),

$$Q(t_*) = \varphi^*(b, z_*, t_*) - \varphi^*(a, z_*, t_*).$$

Here

$$Q(b, t_*) = (h_2 - h_1)^{-1} \int\limits_{h_1}^{h_2} \varphi(b, z_*, t_*)\, dz_*, \qquad Q(a, t_*) = (h_4 - h_3)^{-1} \int\limits_{h_3}^{h_4} \varphi(a, z_*, t_*)\, dz_*$$

($h_1, \ldots, h_4$ are the boundaries of the electroded outer and inner cylindrical surfaces).

4. Numerical Analysis of Results. As an example, we consider a piezoceramic cylinder ($a = 0.5b$ and $h = 1.5b$) made of TsTS-19 composition [1]. Table 1 gives the eigenvalues λ_{in} ($i = \overline{1,3}$) for the first three modes of axisymmetric torsional waves of the cylinder for various number of half-waves along its generatrix ($n = \overline{0,3}$). From Table 1 it follows that the wave propagation velocity depends on the method of measuring the electrical signal [the boundary conditions of types (a) and (b)]. For a device with high input impedance [the boundary conditions of type (a)], the eigenvalues are larger. In this case, in addition, a change in vibration harmonics is observed. For a fixed value of i, an increase in n leads to the occurrence of lower frequencies.

Figure 1 shows a curve of the tangential displacements of the outer curvilinear surface of the cylinder versus time t in the case of application of an electrical load $V(z,t) = V(t) = V_0 \sin \theta t$ at various frequencies of the forced vibrations θ (λ_{10} are the fundamental eigenvalues).

The calculation results show that the assumption of the steady-state forced vibrations can be used only in the case of a significant difference in the frequency characteristics between the external action and the fundamental eigenvalues of the cylinder.

Figure 2 shows a change in the relative displacements along the height of the cylinder ($r = 1$, $\theta = 0.3\lambda_{10}$, and $h = b$) in the case of application of an axisymmetric electrical load on half of the element studied ($h_1 = h_3 = 0$ and $h_2 = h_4 = h/2$) for various thickness of the cylinder walls. From Fig. 2, it follows that the character of change in the quantities studied is significantly affected by a decrease in the thickness of the element.

It should be noted that the algorithm allows solutions for the cylinder to be obtained for different physically realizable electrical and mechanical boundary-value conditions.

REFERENCES

1. V. T. Grinchenko, A. F. Ulitko, and N. A. Shul'ga, *Mechanics of Coupled Fields in Structural Members* [in Russian], Naukova Dumka, Kiev (1989).
2. N. A. Shul'ga and A. M. Bolkisev, *Vibrations of Piezoelectric Solids* [in Russian], Naukova Dumka, Kiev (1990).
3. Yu. É. Senitskii, "Dynamic problem of electroelasticity for an inhomogeneous cylinder," *Prikl. Mat. Mekh.*, **57**, No. 1, 116–122 (1993).
4. I. N. Ermolov, *Ultrasonic Converters for Undestructible Control* [in Russian], Mashinostroenie, Moscow (1986).
5. Yu. É. Senitskii, "Multicomponent generalized finite integral transform and its application to unsteady problems of mechanics, *Izv. Vyssh. Uchebn. Zaved., Mat.*, No. 4, 57–63 (1991).
6. E. Jahnke, F. Emde, and Lösh, *Tafeln Hhërer Funktionen*, Teubuer, Stuttgart (1959).
7. I. E. Tamm, *Foundations of Electricity Theory* [in Russian], Nauka, Moscow (1984).